Abelian Division Fields Over Real Quadratic Fields

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Abstract

An n division field of an elliptic curve is an extension field containing all points of n torsion. It is of interest to find when these fields are abelian. Previously, Enrique González-Jiménez and Álvaro Lozano-Robledo showed what n it is possible to have abelian division fields for elliptic curves defined over $\mathbb{Q}$.
In this project we investigate when abelian division fields of non-CM elliptic curves arise after a base change from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{5})$.

Elliptic Curves

An **elliptic curve**, E, over $\mathbb{Q}$ can be defined by an equation of the form $y^2 = x^3 + Ax + B$, where A,B $\in \mathbb{Q}$ and $\Delta_E = -16(4A^3 + 27B^2) \neq 0$.

The Group Law on an Elliptic Curve

There exists a binary operation $\oplus$ such that $(E(\mathbb{C}), \oplus)$ forms a group with O_E as the identity. This operation is known as the **group law** on the elliptic curve. Its construction is known as the **chord-and-tangent method.**

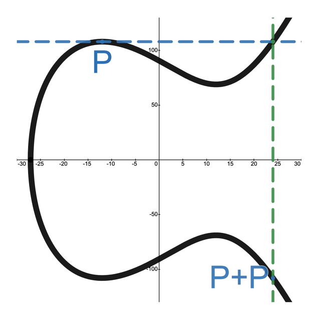


Figure 1:Computation of P+P

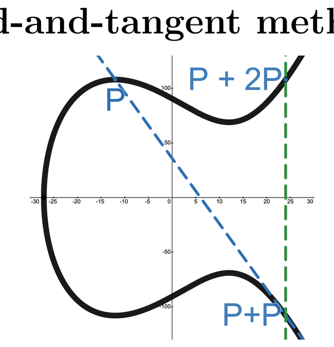


Figure 2:Computation of P+2P

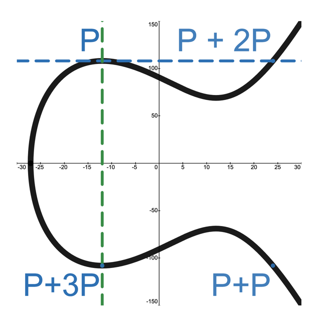


Figure 3:Computation of P+3P

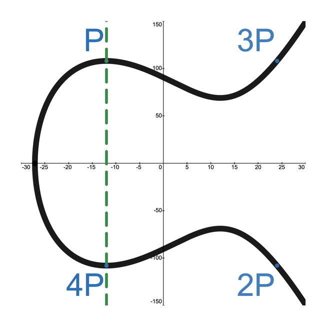


Figure 4:All 5-torsion points and O_E

A point $P \in E(\mathbb{Q})$ has **order** n if n is the smallest positive integer such that $nP = P \oplus P \oplus \cdots \oplus P = O_E$. In no such n exists, P has **infinite order**.
A point $P \in E(\mathbb{Q})$ is called a **torsion point** if it has finite order.

Division Fields

Let E be an elliptic curve. Let K be a field. The **n-th division field of E/K** denoted $K(E[n])/K$ is an extension field of K with all the points of n torsion.
All division fields are Galois extensions. This means $K(E[n])/K$ has a Galois group which fixes K.
A division field is **abelian** if its corresponding Galois group is abelian.

Abelian Division Fields Over $\mathbb{Q}$

- Enrique González-Jiménez and Álvaro Lozano-Robledo previously determined all of the integers n for which there is some elliptic curve $E/\mathbb{Q}$ such that $\mathbb{Q}(E[n])/\mathbb{Q}$ is abelian.
- They proved when $\mathbb{Q}(E[n])$ is as small as possible, that is, when $\mathbb{Q}(E[n]) = \mathbb{Q}(\zeta_n)$, and this is only possible when n = 2,3,4,or 5.
- They were also able to classify all curves such that $\mathbb{Q}(E[n])/\mathbb{Q}$ is an abelian extension and this only happens when n = 2,3,4,5,6, or 8.
- They classified the possible Galois groups that occur for each value of n.
- They also used the Weil pairing theorem to see when $\mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(E[n])$.

Motivating Questions

For what values of n can the n-th division field become abelian over the real quadratic field $\mathbb{Q}(\sqrt{5})$ if it wasn't abelian over $\mathbb{Q}$?

$\mathrm{GL}_2(\mathbb{F}_p)$

The set $E[p]$ of p-torsion points is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. As a result, the set of automorphisms of $E[p]$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_p)$.
Since each field automorphism of the field $\mathbb{Q}(E[p])$ will also be an automorphism of $E[p]$, $\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})$ is isomorphic to a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$.

Method

- We want to start with choosing a prime p.
- We consider all the possible p-division fields for non-CM elliptic curves.
- We determine whether or not $\mathbb{Q}(\sqrt{5})$ can be contained in these division fields.
- If that division field is not abelian over $\mathbb{Q}$, then we want to see if it is abelian over $\mathbb{Q}(\sqrt{5})$.
- We compute Galois groups and their subgroups to determine more about the fields and their subfields.

Narrowing the Possibilities

Proposition: If $\mathbb{Q}(E[n])/\mathbb{Q}$ is abelian then $\mathbb{Q}(\sqrt{5}, E[n])/\mathbb{Q}(\sqrt{5})$ is abelian.

González-Jiménez and Lozano-Robledo tells us when division fields are abelian over $\mathbb{Q}$. The examples of abelian division fields over $\mathbb{Q}$ they have also stay abelian over $\mathbb{Q}(\sqrt{5})$.

Proposition: Let $5 \nmid n$ and $5 \nmid \Delta_E$. If $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ is non-abelian, then $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})(E[n])/\mathbb{Q}(\sqrt{5}))$ is non-abelian as well.

This means that if we want division field to go from non-abelian over $\mathbb{Q}$ to abelian over $\mathbb{Q}(\sqrt{5})$, we want to look at curves where $5 \mid n$ or $5 \mid \Delta_E$.

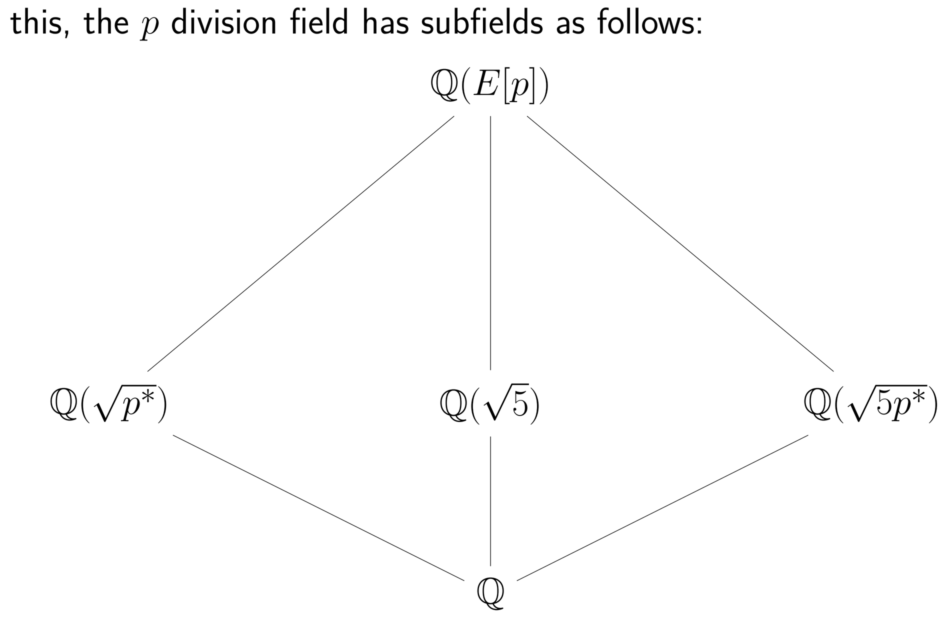
Proposition: If $K(E[n])/K$ is not abelian, then $K(E[dn])/K$ is not abelian for $d \in \mathbb{Z}^+$. And if $K(E[dn])/K$ is abelian, then $K(E[n])/K$ is abelian.

This proposition tells us that if we can say something about prime division fields being non-abelian, then we can say that multiples of those primes produce non-abelian division fields.

Let

$$p^* = \begin{cases} p & \text{if } p \equiv 1 \pmod 4 \\ -p & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

where $p \neq 5$.
With this, the p division field has subfields as follows:



A result by Serre [3] tells us the possibile subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ that $\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})$ can be isomorphic to. This result only holds for **non-CM elliptic curves**.
In the above case, we have 3 degree 2 extensions over $\mathbb{Q}$. This means the corresponding Galois group of the p division field has 3 index 2 subgroups.
When we are in these cases, we only need to consider subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ which have 3 index 2 subgroups.

2 Division Fields

- The 2 division field of an elliptic curve is the field containing the roots of $x^3 + Ax + B$.
- The polynomial $x^3 + Ax + B$ can split in different ways producing different corresponding Galois groups:
  1. All of it's roots could be in $\mathbb{Q}$ and $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \cong \{e\}$.
  2. If it has 1 rational root and 2 irrational roots, $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \cong C_2$.
  3. If the roots are irrational and Δ_E is a perfect square, then $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \cong C_3$.
  4. If the roots are irrational and Δ_E is not a perfect square, then $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \cong S_3$.
- All of those are abelian except for the S_3 case. For this, we have found a result:

Theorem

If the 2 division field is an S_3 extension over $\mathbb{Q}$, then the 2 division field is abelian over $\mathbb{Q}(\sqrt{5})$ iff $\Delta_E = 5d$, where d is a perfect square.

3 Division Fields

- The 3 division field of an elliptic curve is the smallest field containing the 3-torsion points of the elliptic curve.
- $Gal(\mathbb{Q}(E[3])/\mathbb{Q})$ is isomorphic to the following subgroups of $GL_2(\mathbb{F}_3)$: C_2, D_4, D_6, SD_{16}, and S_3.

Theorem

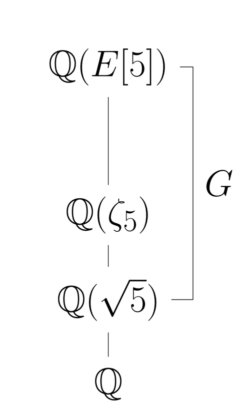
If $Gal(\mathbb{Q}(E[3])/\mathbb{Q})$ is isomorphic to D_6, S_3, or SD_{16}, then $Gal(\mathbb{Q}(E[3])/\mathbb{Q}(\sqrt{5}))$ remains nonabelian.

5 Division Field

- The 5 division field of an elliptic curve is the smallest field containing the 5-torsion points of the elliptic curve.
- $Gal(\mathbb{Q}(E[5])/\mathbb{Q})$ is isomorphic to one of the following: $C_2 \times C_4, C_4^2, OD_{16}, C_4 \wr C_2, C_2 \times F_5, C_{24} : C_2, C_4 \times F_5, C_4, F_5$, or $GL_2(\mathbb{F}_5)$.

Theorem

If $Gal(\mathbb{Q}(E[5])/\mathbb{Q})$ is isomorphic to OD_{16}, then $Gal(\mathbb{Q}(E[5])/\mathbb{Q}(\sqrt{5}))$ remains nonabelian.



7 Division Field

- These groups are subgroups of $\mathrm{GL}_2(\mathbb{F}_7)$ that can appear as Galois groups of 7-division fields over $\mathbb{Q}$ that have at least 3 subgroups of index 2: C_6^2, $C_6 \times S_3$, $C_2 \times F_7$, $C_6 \times D_7$, $C_6 \wr C_2$, $C_6 \times F_7$, and $C_3 \times \mathrm{SD}_{32}$.
- Since C_6^2 is abelian already, we know it will stay abelian after a base change to $\mathbb{Q}(\sqrt{5})$. We have already excluded the others either by some theoretical argument or by computing determinants.
- So far, $C_6 \times S_3$, $C_6 \wr C_2$, and $C_3 \times \mathrm{SD}_{32}$ are the potential index 2 subgroups that could become abelian after a base change to $\mathbb{Q}(\sqrt{5})$.

Future Work

- We plan to finish our work in 3 and 7 division fields. We also plan to follow up on some promising results in 4 and 10 division fields.
- We would like to determine which other primes p and composites n can give us abelian extensions over $\mathbb{Q}(\sqrt{5})$.
- We would like to look at division fields of elliptic curves that are defined over $\mathbb{Q}(\sqrt{5})$ and not over $\mathbb{Q}$.
- We would also like to extend this work to CM elliptic curves as well.

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